arXiv:chao-dyn/9912027v2 6 Aug 2000

Clustering of inertial particles in turbulent flows

E. Balkovsky[a,c], G. Falkovich[b,c], and A. Fouxon[b,c]
[a] School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540 USA
[b] Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel
[c] Institute for Theoretical Physics, UCSB, Santa Barbara, CA 93106, USA

We consider inertial particles suspended in an incompressible turbulent flow. Due to inertia of particles, their velocity field acquires small compressible component. Its presence leads to a new qualitative effect — possibility of clustering. We show that this effect is significant for heavy particles, leading to strong fluctuations of the concentration.

I. INTRODUCTION

Observing air bubbles in water or dust in air, one readily notices that inertial particles suspended in an inhomogeneous flow tend to cluster. For example, such clustering is widely used for flow visualization. Here we develop a statistical theory of this phenomenon. We describe the initial growth of concentration fluctuations and the saturation of that growth due to finite-size effects, imposed either by the Brownian motion or finite distance between the particles. Such theory is supposed to have numerous geophysical and astrophysical applications. A proper account of concentration fluctuations is also necessary for a consistent theory of turbulent suspensions.

Macroscopic description of a dilute suspension can be deduced from the the behavior of a single particle. Consider a small spherical particle with the radius a and the material density ρ_0 suspended in a fluid with the density ρ. The particle's velocity $\boldsymbol{v}$ is related to the fluid velocity $\boldsymbol{u}$ by the equation $d\boldsymbol{v}/dt - \beta d\boldsymbol{u}/dt = (\boldsymbol{u}-\boldsymbol{v})/\tau_s$, where $\beta = 3\rho/(\rho+2\rho_0)$ and $\tau_s = a^2/(3\nu\beta)$ is the Stokes time [1]. Both $\boldsymbol{v}$ and $\boldsymbol{u}$ are evaluated on the particle's trajectory $\boldsymbol{q}(t,\boldsymbol{r})$ that satisfies $\partial_t \boldsymbol{q} = \boldsymbol{v}$ and $\boldsymbol{q}(0,\boldsymbol{r}) = \boldsymbol{r}$. The flow surrounding the particle is assumed to be viscous, which requires $a \ll r_{\rm v}$, where $r_{\rm v}$ is the viscous scale of the flow. This allows one to solve the system for $\boldsymbol{v}$ and $\boldsymbol{q}$ perturbatively in τ_s

$$\boldsymbol{v} = \boldsymbol{u} + (\beta-1)\tau_s[\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\,. \tag{1.1}$$

If such particles are spatially distributed, it is possible to define the particles' velocity field $\boldsymbol{v}(t,\boldsymbol{r})$, which is compressible even if the fluid flow is incompressible [1,2]: $(\nabla\cdot\boldsymbol{v}) = (\beta-1)\tau_s\nabla[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]$. Thus in the above expansion we keep the terms up to the first term with non-vanishing divergence. As we show the smallness of this term may be compensated by large parameters, so that it can lead to significant effects.

The concentration of the particles satisfies the diffusion-advection equation

$$\partial_t n + \nabla(\boldsymbol{v} n) = \kappa\nabla^2 n\,. \tag{1.2}$$

Every particle produces a relative perturbation of the flow that decays as an inverse distance from the particle, i.e. as a/r. Since particles move coherently within the viscous scale $r_{\rm v}$, the condition $a\int_a^{r_{\rm v}} n(r) r^{-1} d^3r \simeq n a r_{\rm v}^2 \ll 1$ has to be satisfied in order to be able to neglect their interaction. This condition is more restrictive than that of a small concentration, $na^3 \ll 1$. If $nar_{\rm v}^2 \ll 1$, the concentration field can be considered as passive, i.e. $\boldsymbol{v}$ is independent of n in Eq. (1.2).

In statistically steady flows, velocity $\boldsymbol{v}$ in Eq. (1.2) must be considered as a random field with a stationary statistics. Evolution of an arbitrary initial condition $n(0,\boldsymbol{r})$ according to Eq. (1.2) ultimately results in the steady state of the concentration fluctuations. Making the decomposition $n(0,\boldsymbol{r}) = n_0 + \delta n(\boldsymbol{r})$, where n_0 is the spatial average of $n(0,\boldsymbol{r})$, one can write the solution in the form

$$n(t,\boldsymbol{r}) = n_0\int d\boldsymbol{r}'\, G(t,\boldsymbol{r},\boldsymbol{r}') + \int d\boldsymbol{r}'\, G(t,\boldsymbol{r},\boldsymbol{r}')\delta n(\boldsymbol{r}') \tag{1.3}$$

We note that the Green's function, G, is nonnegative. At $\kappa = 0$ it is concentrated on the Lagrangian trajectory that passes through the observation point, $\boldsymbol{r}$, at time t. At non-zero κ, the Green's function is non-zero in some region around the Lagrangian trajectory. The size of that support region grows in time due to the combined action of the velocity and diffusion. As long as that size is smaller than the correlation length of the concentration, δn can be taken out of the integral. Then the expectation value of n depends on the ratio of n_0 and the strength of initial fluctuations δn. At large times when the support of the Green's function becomes much larger than the correlation length, the second term contains contributions that cancel each other. The expectation value of n is then determined by the first term. Even though the second term in Eq. (1.3) may grow at these times, it is much smaller than the first term and only gives subleading dependencies. Thus the initial inhomogeneities of the concentration field are irrelevant in studying the long-time evolution and the steady state. Therefore, we will consider a uniform initial concentration below. We will show that evolution of the uniform initial concentration distribution exemplifies most strikingly the inadequacy of the mean field picture for the problem. We choose the units so that $n_0 = 1$.

The paper is organized as follows. First, we analyze the initial period of growth when one can set $\kappa = 0$ (to

be referred to as the ideal case). We show that at this stage of evolution the moments of the concentration grow exponentially for quite an arbitrary velocity statistics (in case of time-decorrelated velocity this has been shown in [3]). The next section is devoted to the analysis of larger times. We show that diffusion modifies the growth of the moments and finally brings about saturation.

II. THE IDEAL CASE

The diffusivity of macroscopic particles is usually very small so that we will be interested in the statistics of n in the limit of small κ. Starting with a uniform distribution one expects that at moderate times the diffusion term can be neglected until very thin structures are developed. Let us consider this period of evolution.

To find the concentration $n(t_1, \boldsymbol{r}_1)$ one has to count all the particles that come to a small volume (still, containing many particles) around $\boldsymbol{r} = \boldsymbol{r}_1$ and divide the result by this volume. To find which particles comes to this volume one has to track the trajectories of the particles backwards in time to $t = 0$. Particles perform combined Lagrangian and Brownian motions. At $t \approx t_1$, the size l of the region occupied by particles grows as $l^2(t) \sim \kappa(t_1 - t)$. Velocity gradient λ produces another mechanism of size stretching which takes over when $l \gtrsim \sqrt{\kappa/\lambda}$. Smallness of diffusion means that the Schmidt number, $\mathrm{Sc} \equiv \nu/\kappa$ is large so that $\sqrt{\kappa/\lambda} \ll r_{\mathrm{v}}$. As one goes further backwards in time the impact of diffusion on particles' motion becomes negligible. The time needed for l to reach the diffusive scale $\sqrt{\kappa/\lambda}$ is of the order of $1/\lambda$. On a larger time-scale the role of diffusion is to create an initial volume of finite size $\sqrt{\kappa/\lambda}$. In other words, diffusion introduces the smallest scale into the problem, so that fluid particles cannot be localized on the distances smaller than this scale. To find the concentration at a point in the ideal case one should track back an infinitesimal volume around that point. However, in the case of a nonzero diffusivity one should take the volume with a finite size $\sqrt{\kappa/\lambda}$. Since $\sqrt{\kappa/\lambda}$ can be smaller than the minimal scale of coarse-graining $n^{-1/3}$ we introduce r_{d}, which is given by the largest of two scales: $\sqrt{\kappa/\lambda}$ and $n^{-1/3}$. To summarize, $n(t_1, \boldsymbol{r}_1)$ is given by the relative change of the volume of the size r_{d} taken around the point $\boldsymbol{r}_1$ and tracked backwards in time along the Lagrangian trajectory. Let us note that the velocity gradient fluctuates in a random flow and so does the diffusion scale. We neglect these fluctuations (c.f. [4]) because they do not change the dependence of the concentration on large parameters, which are either time in the transient regime or Reynolds and Schmidt numbers in the steady state.

At $t \lesssim \lambda^{-1}\ln(r_{\mathrm{v}}/r_{\mathrm{d}})$ the r_{d}-volume (the region that acquires the size r_{d} at $t = t_1$) always stays within the viscous scale and hence evolves in a uniform velocity gradient. Its relative change is the same as for an infinitesimal volume, which means that the concentration behaves the same as in the ideal case. Equation (1.2) written in the Lagrangian frame therefore becomes the ordinary differential equation $dn/dt = -n(\nabla \cdot \boldsymbol{v})$. Here $(\nabla \cdot \boldsymbol{v})$ is a function of time, which fluctuates in a random flow. If the Lagrangian correlation time of the fluid velocity, $\boldsymbol{u}$, is finite, which is true for most flows of geophysical interest, then $(\nabla \cdot \boldsymbol{v}) \propto \nabla[(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}]$ has also a finite correlation time, τ. At $t \gg \tau$, the concentration logarithm, $X(t) \equiv \ln[n(t)/n(0)] = -\int_0^t (\nabla \cdot \boldsymbol{v}) dt'$, is a sum of a large number of random variables. The theory of large deviations assures that the probability density function (PDF) has the form $\mathcal{P}(X) \propto \exp[-ts(X/t)]$, where s is a non-negative convex function [5]. To calculate the moments of the concentration in the Eulerian frame one has to take every Lagrangian element with its own weight proportional to its volume, i.e. to the inverse concentration (see Eqs. (2.20) and (2.21) below). We obtain

$$\langle n^\alpha(t, \boldsymbol{r}) \rangle \propto \int dX \exp[(\alpha - 1)X - ts(X/t)]\,. \tag{2.1}$$

At large times, this integral can be found using the saddle-point approximation. The saddle-point X_α is given by $s'(X_\alpha/t) = \alpha - 1$, which implies $X_\alpha \propto t$. Hence the moments generally behave exponentially in time: $\langle n^\alpha(t) \rangle \propto \exp(-\gamma(\alpha)t)$.

Let us show that the conclusion on exponential behavior of moments is enough to establish the most interesting properties of this stage of evolution. The number of particles is conserved, i.e. $\langle n \rangle$ is time- independent. Hence $\gamma(1) = 0$. It is also obvious that $\gamma(0) = 1$. Due to the Hölder inequality, the function $\gamma(\alpha)$ is convex. Therefore, $\gamma(\alpha)$ is positive for $0 < \alpha < 1$ and negative otherwise. Low-order moments decay whereas high-order and negative moments grow. The decay rate is $\langle \log |n| \rangle / t = -d\gamma(\alpha)/d\alpha|_{\alpha=0} < 0$, i.e. n decays almost everywhere. Since the mean concentration is conserved, n has to grow in some (smaller and smaller) regions, which implies growth of high moments. The growth of passive scalar fluctuations in the particular case of a short-correlated compressible flow was described in [3].

The finite value of the coarse-grained volume comes into play at $t \gtrsim \lambda^{-1}\ln(r_{\mathrm{v}}/r_{\mathrm{d}})$. Indeed, since particles separates backwards in time, the size of the volume exceeds r_{v} at $t = 0$. In other words, spots originated from different viscous domains come into contact at $t = t_1$. A careful analysis of the time-span of the ideal case approximation demands more detailed information on the Lagrangian dynamics at scales smaller than r_{v} (later referred to as small-scale Lagrangian dynamics). It is discussed in subsection II A.

A. Small-scale Lagrangian dynamics of compressible fluids with finite Lagrangian correlation time

Let us present general analysis of Lagrangian statistics assuming that the distances between particles are smaller than $r_{\rm v}$. Consider two Lagrangian trajectories $\boldsymbol{q}(t,\boldsymbol{r}_1)$ and $\boldsymbol{q}(t,\boldsymbol{r}_2)$, satisfying the equations $\partial_t\boldsymbol{q}(t,\boldsymbol{r}_i)=\boldsymbol{v}(t,\boldsymbol{q}(t,\boldsymbol{r}_i))$ and the initial conditions $\boldsymbol{q}(0,\boldsymbol{r}_i)=\boldsymbol{r}_i$. The distance between the two trajectories $\boldsymbol{R}=\boldsymbol{q}(t,\boldsymbol{r}_1)-\boldsymbol{q}(t,\boldsymbol{r}_2)$ satisfies the equation $\partial_t\boldsymbol{R}=\boldsymbol{v}(t,\boldsymbol{q}(t,\boldsymbol{r}_1))-\boldsymbol{v}(t,\boldsymbol{q}(t,\boldsymbol{r}_2))\approx\sigma\boldsymbol{R}$, when R is much smaller than the viscous length and the velocity difference can be approximated by the first term in its Taylor expansion. The rate-of-strain matrix,

$$\sigma_{\alpha\beta}(t)=\left.\frac{\partial v_\alpha(t,\boldsymbol{r})}{\partial r_\beta}\right|_{\boldsymbol{r}=\boldsymbol{q}(t,\boldsymbol{r}_2)} \tag{2.2}$$

is what determines the deformation of a fluid blob of a small size. If the velocity statistics is spatially homogeneous, then the uniform sweeping is statistically irrelevant and only the deformation part of the Lagrangian transformation is significant. It is given by $\boldsymbol{R}=W\boldsymbol{R}_0$, where the evolution matrix W satisfies

$$\partial_t W=\sigma W\,,\quad W|_{t=0}=1\ .$$

The Lagrangian evolution may be thus considered as a linear mapping given by the affine transformation W, whose statistics is determined by the statistics of σ. Such Lagrangian mapping can be described universally at times much larger than the correlation time, τ, of σ. In the one-dimensional case one finds $\ln W\equiv\rho=\int_0^t dt'\sigma(t)$, so that at $t\gg\tau$ one deals with a sum of large number of independent random variables. The probability distribution function (PDF) of ρ is completely determined by the entropy function S (see e.g. Ref. [5])

$$\mathcal{P}(t,\rho)=\frac{1}{Z(t)}\exp\left[-tS\left(\frac{\rho-\lambda t}{t}\right)\right], \tag{2.3}$$

where λ is the average value $\langle\sigma\rangle$ and $1/Z$ is the normalization factor. At $\rho=\lambda t$ the PDF has a sharp maximum, described by the central limit theorem, where $S(x)\approx x^2/(2C)$. Here C is the dispersion of σ:

$$C=\int dt'\langle\langle\sigma(t)\sigma(t')\rangle\rangle\,.$$

This analysis can be generalized to higher dimensions. The idea is briefly presented here following the recent exposition in [6]. At large times, the Lagrangian transformation can be represented as a stretching or contraction along fixed orthogonal directions followed by a rotation. Indeed, one can represent W as the product $M\Lambda N$, where M and N are orthogonal matrices, and Λ is a diagonal matrix [7,8]. At large times the matrix N becomes asymptotically constant, as follows from the Oseledec theorem for W^tW [9]. Excluding the constant matrix N by the proper choice of initial basis one finds $W=M\Lambda$. This representation shows that in the frame rotating with the fluid blob the Lagrangian mapping is just a stretching along fixed directions. Stretching accumulates so that the characteristic time of change of the matrix Λ is t while the matrix M changes on the much shorter time-scale which is of order of the inverse elements of σ. Since the time scales of M and Λ are widely different, the matrices are statistically independent. Indeed, changing σ at the last stage of evolution with duration τ one changes M completely, whereas Λ is changed by the amount of the order $\tau/t\ll 1$. It means that fixing the value of $\Lambda(t)$ does not change the distribution of $M(t)$. The matrix M is uniformly distributed over the rotation group. The PDF of the eigen values of the matrix $\Lambda=diag\,[\exp(\rho_1),\dots,\exp(\rho_d)]$, is given by

$$\mathcal{P}(t,\rho_i)=\frac{1}{Z}\exp\left[-tS\left(\frac{\rho_1-\lambda_1 t}{t},\dots,\frac{\rho_d-\lambda_d t}{t}\right)\right]$$
$$\times\theta(\rho_1-\rho_2)...\theta(\rho_{d-1}-\rho_d)\,. \tag{2.4}$$

Here θ is the step function that orders the eigen values, so that $\rho_1\geq\rho_2\geq..\geq\rho_d$. The constants λ_i (ordered in the same way) are the Lyapunov exponents of the flow. In principle, they can be expressed via the statistics of σ (see e.g. [6,7]). We will assume that the Lyapunov spectrum is non-degenerate, i.e. $\lambda_1>\lambda_2>\ldots>\lambda_d$. The normalization factor Z in Eq. (2.4) is a function of time. Equation (2.4) shows that at times much larger than the correlation time of σ, independently of the statistics of σ, the statistics of W is characterized by a single function S of d variables. This entropy function is convex and positive, and it has the expansion

$$S=x_iC_{ij}^{-1}x_j \tag{2.5}$$

near its minimum at $x_1=\ldots=x_d=0$. Here C_{ij} is the covariance matrix of σ. Note that we assume the entropy function to be nonzero at least in some interval of ρ_i which physically means that the flow is random. We also assume that the flow is compressible so that S is a regular function of all variables (the distribution in the incompressible case is obtained by a limiting operation). The homogeneous dependence of the PDF on ρ_i and t is often sufficient to establish universal statistical laws independently of the details of velocity statistics [6,10,11].

At large times, $\mathcal{P}(t,\rho_i)$ has a sharp maximum at $\rho_i=\lambda_i t$. The existence of generally non-zero Lyapunov exponents manifests itself in various growth rates. For example, the average logarithmic rate of separation of two particles located within the viscous scale of the flow is given by λ_1, the corresponding growth rate of the fluid volume is $\sum_{i=1}^d\lambda_i$ etc.

For an incompressible random flow $\lambda_1>0$ [12], while the average value of σ is zero $\langle\sigma\rangle=0$. The appearance

of non-zero λ_1 is related to the interplay between rotational and stretching degrees of freedom [13,6,12] (for a scalar equation one would have $\lambda_1 = \langle\sigma\rangle = 0$). The simplest way to appreciate the existence of positive Lyapunov exponent is to consider (following [12]) an example of a saddle-point 2d flow $v_x = \lambda x, v_y = -\lambda y$ where the stretching directions satisfy $\cos\phi \geq (1+\lambda^2)^{-1/2}$ that is their measure is larger than $1/2$.

Compressibility introduces another mechanism of correlations affecting the stretching. There are more Lagrangian particles in the contracting regions with $\nabla\cdot\boldsymbol{v} < 0$, leading to to the appearance of negative average gradients in the Lagrangian frame. By isotropy one has $d\langle\sigma_{\alpha\beta}\rangle = \delta_{\alpha\beta}\langle\nabla\cdot\boldsymbol{v}(t,\boldsymbol{r})|_{\boldsymbol{r}=\boldsymbol{q}(t,\boldsymbol{r}_0)}\rangle$. Averaging over the volume, one obtains

$$\begin{aligned}\langle\nabla\cdot\boldsymbol{v}(t,\boldsymbol{r})|_{\boldsymbol{r}=\boldsymbol{q}(t,\boldsymbol{r}_0)}\rangle &= \int\frac{d\boldsymbol{r}_0}{V}\,\nabla\cdot\boldsymbol{v}(t,\boldsymbol{r})|_{\boldsymbol{r}=\boldsymbol{q}(t,\boldsymbol{r}_0)}\\ &= \int\frac{d\boldsymbol{r}}{V}\nabla\cdot\boldsymbol{v}(t,\boldsymbol{r})\left(\det\left|\left|\frac{\partial\boldsymbol{q}(t,\boldsymbol{r}_0)}{\partial r_0}\right|\right|\right)^{-1}\Bigg|_{\boldsymbol{q}(t,\boldsymbol{r}_0)=\boldsymbol{r}}\\ &= \left\langle\nabla\cdot\boldsymbol{v}(t,\boldsymbol{r})\left(\det\left|\left|\frac{\partial\boldsymbol{q}(t,\boldsymbol{r}_0)}{\partial r_0}\right|\right|\right)^{-1}\Bigg|_{\boldsymbol{q}(t,\boldsymbol{r}_0)=\boldsymbol{r}}\right\rangle, \qquad (2.6)\end{aligned}$$

We observe that this Lagrangian average generally coincides with the Eulerian average $\int d\boldsymbol{r}\nabla\cdot\boldsymbol{v}(t,\boldsymbol{r})/V$ only in the incompressible case, where it is zero. In the compressible flow, the integral (2.6) is zero at zero time (when we set the initial conditions for the Lagrangian trajectories so the measure is uniform back then). The average is getting time-independent and negative at times larger than the velocity correlation time when the compressed regions with negative $\boldsymbol{\nabla}\cdot\boldsymbol{v}$ acquire higher weight than the expanded ones. Let us illustrate this conclusion by considering the physically interesting case of $\nabla\cdot\boldsymbol{v}$ short-correlated in time (to be discussed in much detail below). Taking t in (2.6) larger than the correlation time of $\nabla\cdot\boldsymbol{v}$ yet small enough to allow for the expansion

$$\det{}^{-1}\left|\left|\frac{\partial\boldsymbol{q}(t,\boldsymbol{r})}{\partial\boldsymbol{r}}\right|\right| \approx 1 - \int_0^t\nabla\cdot\boldsymbol{v}(t',\boldsymbol{r})dt'\,,$$

one finds

$$\begin{aligned}&\langle\nabla\cdot\boldsymbol{v}(t,\boldsymbol{r})|_{\boldsymbol{r}=\boldsymbol{q}(t,\boldsymbol{r}_0)}\rangle =\\ &-\frac{1}{2}\int_0^t dt'\langle\nabla\cdot\boldsymbol{v}(t,\boldsymbol{r})\nabla\cdot\boldsymbol{v}(t',\boldsymbol{r})\rangle\,.\end{aligned}$$

Negative $\langle tr\sigma\rangle = \nabla\cdot\boldsymbol{v}$ leads to a suppression of the stretching by the velocity field. If one decomposes σ into "incompressible and compressible parts"

$$\sigma_{\alpha\beta} = \left(\sigma_{\alpha\beta} - \frac{1}{d}\delta_{\alpha\beta}\,tr\sigma\right) + \frac{1}{d}\delta_{\alpha\beta}\,tr\sigma\,,$$

then from the explicit expressions for Lyapunov exponents [6,7] it is easy to find that the Lyapunov exponents of the incompressible process $\sigma_{\alpha\beta} - \delta_{\alpha\beta}tr\sigma/d$ (with $\lambda_1 > 0$) get uniformly shifted down by $\langle tr\sigma\rangle/d$. At a sufficient degree of compressibility, all the exponents may become negative (in the one-dimensional case where compressibility is maximal one can prove that $\lambda < 0$, see below).

Let us stress the difference in Eulerian and Lagrangian averages appearing in the compressible case. An Eulerian average is uniform over the space while in a Lagrangian average every trajectory comes with its own weight determined by the local rate of volume change. This difference is of an utmost importance in the discussion of backward in time Lagrangian statistics to which we pass now.

We have seen that to find the concentration we must consider the evolution of a fluid blob backwards rather than forward in time. This is a general situation: to find the value of an advected field at the given space-time point, $\boldsymbol{r}, T$, one should consider the Lagrangian trajectory $\boldsymbol{q}(t|T,\boldsymbol{r})$ fixed by its final (rather than initial) position:

$$\partial_t\boldsymbol{q}(t|T,\boldsymbol{r}) = \boldsymbol{v}(t,\boldsymbol{q}(t|T,\boldsymbol{r})),\quad \boldsymbol{q}(T|T,\boldsymbol{r}) = \boldsymbol{r}.$$

The initial point $\boldsymbol{q}(0|T,\boldsymbol{r})$ depends on the velocity realization, so that it is random and not fixed as in the above analysis. We denote the Lagrangian quantities related to the trajectories fixed by their destination by the tilde sign. The strain matrix is defined as

$$\tilde\sigma_{\alpha\beta} = \left.\frac{\partial v_\alpha(t,\boldsymbol{r})}{\partial r_\beta}\right|_{\boldsymbol{r}=q(t|T,\boldsymbol{r}_0)}. \qquad (2.7)$$

This must be compared with the definition of σ, where the initial condition fixes the Lagrangian trajectory. The matrices σ and $\tilde\sigma$ generally have different statistical properties. The evolution matrix $\tilde W$ is now defined by

$$\partial_t\tilde W(t|T,\boldsymbol{r}) = \tilde\sigma\tilde W(t|T,\boldsymbol{r}),\quad \tilde W(0|T,\boldsymbol{r}) = 1\,.$$

Its value at $t = T$ determines the deformation of fluid blobs coming to a fixed final point. Let us relate $\tilde W$ to W which we now write with a spatial argument

$$W_{ij}(t|t',\boldsymbol{r}) = \frac{\partial q_i(t|t',\boldsymbol{r})}{\partial r_j}.$$

The expression for $\tilde W(t|T,\boldsymbol{r})$ in terms of $W(t|t',\boldsymbol{r})$ is given by $\tilde W(t|T,\boldsymbol{r}) = W(t|T,\boldsymbol{r})W^{-1}(0|T,\boldsymbol{r})$, so that $\tilde W(T) = W^{-1}(0|T,\boldsymbol{r}_0)$. Differentiating the identity $\boldsymbol{q}(T|0,\boldsymbol{q}(0|T,\boldsymbol{r})) = \boldsymbol{r}$ one finds $W(T|0,\boldsymbol{q}(0|T,\boldsymbol{r})) = W^{-1}(0|T,\boldsymbol{r})$, which relates W and $\tilde W$:

$$\tilde W(T|T,\boldsymbol{r}) = W(T|0,\boldsymbol{q}(0|T,\boldsymbol{r}))\,.$$

To express the statistics of $\tilde W$ in terms of the Lagrangian characteristics we use the same transformation that we used for transforming the average of $\nabla\cdot\boldsymbol{v}$ in the

Lagrangian frame to the usual Eulerian average. Namely, for the average over the volume of the flow of any function f one has

$$
\begin{aligned}
\langle f\{\tilde{W}(T)\}\rangle &= \int \frac{d\boldsymbol{r}_0}{V} f\{W(T|0,\boldsymbol{q}(0|T,\boldsymbol{r}_0))\} \\
&= \int \frac{d\boldsymbol{x}}{V} f\{W(T|0,\boldsymbol{x})\}\det\left|\left|\frac{\partial \boldsymbol{q}(T|0,\boldsymbol{x})}{\partial x}\right|\right| \\
&= \langle f\,\{W(T|0,\boldsymbol{x})\}\det W(T|0,\boldsymbol{x})\rangle\ . \qquad (2.8)
\end{aligned}
$$

Again, Lagrangian and Eulerian averages coincide for incompressible flow, when $\det W \equiv 1$. In general, it is necessary to account for the local volume change when passing from one average to another. Since $\det W = \exp\sum\rho_i$, then considering passive fields one has to take the following probability distribution of stretching/contraction eigen values

$$
\tilde{\mathcal{P}}(t,\rho_i) = \frac{1}{Z}\exp\left[\sum_{i=1}^{d}\rho_i - tS\left(\frac{\rho_1-\lambda_1 t}{t},\ldots,\frac{\rho_d-\lambda_d t}{t}\right)\right] \times\theta(\rho_1-\rho_2)\ldots\theta(\rho_{d-1}-\rho_d)\ . \qquad (2.9)
$$

Here S and Z are the same as in (2.4) that is what one can measure in studying (forward-in-time) particle dispersion. Note that the correct normalization of $\tilde{\mathcal{P}}$ is guaranteed by the volume conservation $\langle\det W\rangle = 1$ following from (2.8) with $f=1$.

The Lyapunov exponents $\tilde{\lambda}_i$ of $\tilde{W}$ are determined by the extremum of the exponent:

$$
\tilde{\lambda}_i = \lambda_i + y_i, \qquad (2.10)
$$

where y_i are determined from $\partial S(y_1,..,y_d)/\partial y_i = 1$. An important remark is that y_i cannot generally be expressed via the Lyapunov exponents λ_i only. They depend on the form of the entropy function S and hence on the details of velocity statistics. Indeed, every trajectory comes with its own weight determined by the local rate of volume change. The consequence is that passive fields behavior in a compressible flow does not enjoy the same degree of universality as in the incompressible case (when, for instance, the growth rate of the magnetic fluctuations is determined solely by the spectrum of the Lyapunov exponents λ_i and is independent of the form of the entropy function [10]).

Even though we will use only the properties of the matrix $\tilde{W}$, we also mention the probability distribution function of the eigen values of matrix $W(0|T,\boldsymbol{r})$ which directly determines the evolution backwards in time

$$
\tilde{\mathcal{P}}(t,\rho_i) = \frac{1}{Z}\exp\left[-\sum_{i=1}^{d}\rho_i - tS\left(\frac{-\rho_1-\lambda_d t}{t},\ldots,\frac{-\rho_d-\lambda_1 t}{t}\right)\right]\theta(\rho_1-\rho_2)\ldots\theta(\rho_{d-1}-\rho_d)\ .
$$

which follows from the above results using $W(0|T,\boldsymbol{r}) = \tilde{W}^{-1}(T|T,\boldsymbol{r})$. It follows that the backward-in-time Lyapunov exponents are given by $-\tilde{\lambda}_i$ and not by the naive guess $-\lambda_i$, which holds only in the incompressible case. In particular, particles diverge backwards in time with exponent $-\tilde{\lambda}_d$.

The difference between Lyapunov exponents λ_i and $\tilde{\lambda}_i$ can be illustrated in the one-dimensional case where their signs are definite and opposite. Indeed, the conservation of the total fluid volume together with spatial homogeneity imply

$$
\langle W\rangle = \frac{1}{Z}\int d\rho\exp\left[\rho - tS\left(\frac{\rho-\lambda t}{t}\right)\right] = 1. \qquad (2.11)
$$

Let us consider this identity at large times when the integral can be calculated by the saddle-point method. First, we note that at large t the normalization factor $Z = \int d\rho\exp(-tS)$ is determined by the region, where $S(x)\propto x^2$ so that $Z\propto t^{1/2}$. Next, one can rewrite the integral (2.11) as $tZ^{-1}\int dx\exp[t(\lambda+x-S(x))]$. It is determined by the point x_* where $x-S(x)$ is maximal. Since $x-S(x)$ takes positive values near $x=0$ we conclude that $x_*-S(x_*)>0$. Therefore $\lambda = S(x_*)-x_*<0$, for the integral (2.11) to be time-independent.

On the contrary, $\tilde{\lambda}$ is positive. Defining $\rho + S((\rho-\lambda t)/t) \equiv \tilde{S}((\rho-\tilde{\lambda}t)/t)$ one has the condition $\int d\rho\exp[-\rho-t\tilde{S}(\rho/t-\tilde{\lambda})] = 1$ which gives $\tilde{\lambda}>0$.

The above results are generalized to higher dimensions as follows. Since we consider the flow to be contained in a fixed volume then $\langle\det W\rangle = 1$, and in the same manner one finds that the mean logarithmic rate-of-change of the volume elements $\sum\lambda_i\le 0$ (which, in particular, implies $\lambda_3\le 0$). From the explicit expressions for λ_i [6,7] one finds $\sum\lambda_i = \langle\nabla\cdot\boldsymbol{v}\rangle$, so that we have proved that the Lagrangian average $\langle\nabla\cdot\boldsymbol{v}\rangle$ is nonpositive, the result stated above on physical grounds. The corresponding inequality on $\tilde{\lambda}_i$ is $\sum\tilde{\lambda}_i\ge 0$ (implying $\tilde{\lambda}_1\ge 0$). We arrive at a somewhat surprising conclusion that for the Lagrangian dynamics one has average compression of volumes, whereas passive fields rather feel average expansion. The physical meaning of this effect is transparent: as we go away (either forward in calculating λ_i or backwards in calculating $\tilde{\lambda}_i$) from the moment where we imposed a uniform Lagrangian measure, the volume rate-of change is getting negative in a fluctuating compressible flow. To avoid misunderstanding, let us stress that for a physical quantity $x(t)$ (volume of a fluid element in this case) the conservation of the mean value $\langle x(t)\rangle$ does not contradict to a nonzero rate of change $t^{-1}\langle\ln x(t)\rangle$.

The general considerations can be illustrated using a particular case of the velocity statistics, the Kraichnan model of a short-correlated Gaussian velocity with the variance

$$
\langle v_\alpha(t,\boldsymbol{r})v_\beta(0,0)\rangle = [V_0\delta_{\alpha\beta} - \mathcal{K}_{\alpha\beta}(\boldsymbol{r})]\,\delta(t)\,, \qquad (2.12)
$$

$$
\mathcal{K}_{\alpha\beta} = D\left[(d+1-2\Gamma)\delta_{\alpha\beta}r^2 + 2(d\Gamma-1)r_\alpha r_\beta\right]\ . \qquad (2.13)
$$

Here Γ is the ratio of the variances of $\nabla_\alpha v_\alpha$ and $|\boldsymbol{\nabla v}|$ respectively, it is thus the degree of compressibility that may vary between 0 and 1. The quadratic dependence of the correlation function on the coordinate corresponds to the expansion of the velocity difference we made above. For a velocity defined by Eqs. (2.12) and (2.13), a straightforward calculation gives

$$\lambda_i/D = d(d+1-2i) - 2\Gamma[d+(d-2)i]\ . \qquad (2.14)$$

In the incompressible case, $\Gamma = 0$, this formula has been derived in [14]. For a general compressible case, λ_1 has been derived in [4], where it has been also observed that λ_1 changes sign at $\Gamma = d/4$. The entropy function has the form (2.5) for arbitrary values of x. One can also find

$$C_{ij} = 4D\{[d+\Gamma(d-2)]\delta_{ij} - 1 + \Gamma d\}\ .$$

We see from (2.14) that compressibility indeed diminishes the Lyapunov exponents. It is interesting to compare this with the influence of compressibility on Lagrangian dynamics in a multiscale velocity: there, the Lagrangian trajectories either explosively separate or implosively collapse depending on whether the degree of compressibility is small or large respectively [15].

The Lyapunov exponents $\tilde\lambda_i$ that govern the behavior of the passive fields are enhanced by compressibility since y_i are positive (see Eq. (2.10)). For the Kraichnan model one has $y_i = \sum_j C_{ij}/2$, so that

$$\tilde\lambda_i/D = d(d+1-2i) + 2\Gamma[d^2-(d-2)i-2]\ . \qquad (2.15)$$

For $d = 2, 3$ one has

$$\tilde\lambda_1 = 2D(1+2\Gamma)\,, \quad \tilde\lambda_2 = -2D(1-2\Gamma)\,,$$
$$\tilde\lambda_1 = 6D(1+2\Gamma)\,, \quad \tilde\lambda_2 = 10D\Gamma\,, \quad \tilde\lambda_3/ = -2D(3-4\Gamma)\,.$$

The compressibility, Γ, is identically equal to unity in $d = 1$, where instead of Eq. (2.13) one should write $\mathcal{K} = Dx^2$. Then, Eqs. (2.14) and (2.15) are replaced by $\lambda = -D$ and $\tilde\lambda = D$ respectively.

Comparison of (2.14) and (2.15) shows that $\lambda_i = -\tilde\lambda_{d+1-i}$. This relation is due to time-reversibility of the short-correlated velocity. In particular, λ_1 and $\tilde\lambda_d$ change sign at the same degree of compressibility $\Gamma = d/4$. This is peculiar for a short-correlated case and does not hold for an arbitrary velocity statistics. In other words, the change of the regime from stretching to contraction in the forward Lagrangian dynamics does not generally correspond to the change of the regime in the passive fields, which are related to the backwards in time Lagrangian dynamics.

B. Applicability of the ideal case approximation

Let us now determine the domain of validity of the ideal case approximation. Our starting point will be the dynamic expression for the Green's function that can be derived explicitly in this case. To find when the concentration starts to be determined by the particles that were initially separated by a distance larger than the viscous scale, we must analyze the support of the Green's function, $G(t,\boldsymbol{r}|t'=0,\boldsymbol{r}')$, as a function of the initial coordinate, $\boldsymbol{r}'$. At $\kappa = 0$ the dynamics is purely Lagrangian, so that

$$G(t,\boldsymbol{r}|0,\boldsymbol{r}') = \frac{1}{\det \tilde W(t,\boldsymbol{r})}\delta(\boldsymbol{r}' - \boldsymbol{q}(0|t,\boldsymbol{r}))\,.$$

This formula has a clear meaning: the change of the concentration at a point is completely determined by the volume compression factor along the Lagrangian trajectory. Small diffusion is equivalent to adding a Brownian motion to the velocity. It leads to a smearing of the region around the Lagrangian trajectory from which particles come to the observation point. As a function of the initial arguments, the Green's function satisfies the Hermite-conjugate evolution equation

$$\partial_{t'}G + (\boldsymbol{v}(t',\boldsymbol{r}'),\nabla_{r'})G = -\kappa\nabla^2_{r'}G\,. \qquad (2.16)$$

As long as the support of G is much smaller than the viscous scale, one can expand the velocity in the vicinity of the Lagrangian trajectory in the Taylor series. The homogeneous component is excluded by passing to the moving frame and the first non-trivial term contains $\tilde\sigma$:

$$\partial_{t'}G + \tilde\sigma_{\alpha\beta}(t')r'_\beta\nabla_\alpha G = -\kappa\nabla^2_{r'}G\,. \qquad (2.17)$$

This equation can be solved in the Fourier space

$$\begin{aligned} G(t,\boldsymbol{r}|0,\boldsymbol{r}') &= \frac{1}{\det \tilde W(t,\boldsymbol{r})} \\ &\times \int \frac{d\boldsymbol{k}}{(2\pi)^d}\exp\left[i\boldsymbol{k}\cdot[\boldsymbol{r}'-\boldsymbol{q}(0|t,\boldsymbol{r})] - \frac{k^tIk}{2}\right]\,, \qquad (2.18) \\ I &= 2\kappa\int_0^t dt'\,\tilde W^{-1}(t'|t,\boldsymbol{r})\tilde W^{-1,t}(t'|t,\boldsymbol{r})\,. \end{aligned}$$

The matrix I is the inertia tensor of a patch of particles, evaluated at $t = 0$, provided the patch is a sphere with the center at the point $\boldsymbol{r}$ at time t. The particles perform independent Brownian motions together with the Lagrangian motion in the same velocity field (cf. [6]). Let us stress that no averaging has been performed in Eq. (2.18) and therefore the expression for G is purely dynamical. We observe that the size of the region which makes the main contribution to the concentration at a point grows as the largest eigen value of the matrix I, i.e. the square of the linear size grows as $\kappa\int_0^t dt'\exp[-2\tilde\rho_d(t')]$. Since the diffusionless consideration is valid as long as the largest size of the r_d-volume is smaller than the viscous scale, the applicability condition of Eq. (2.18) is

$$\kappa\int_0^t dt'\exp[-2\tilde{\rho}_d(t')]\ll r_{\rm v}^2\,. \qquad (2.19)$$

Below, we will refer to the configurations of velocity with decreasing $\tilde{\rho}_d(t)$ as the contracting configurations. Indeed, for such configurations, particles at the observation point are brought together from larger regions. On the other hand, for the configurations with increasing $\tilde{\rho}_d(t)$, the concentration is determined by the particles initially belonging to the region of the size of the order $\sqrt{\kappa/\lambda}$. Such configurations can be called diverging because the particles in the vicinity of $\boldsymbol{q}(0|t,\boldsymbol{r})$ diverge exponentially.

Formula (2.18) gives $n(t,\boldsymbol{r})=\int G(t,\boldsymbol{r}|0,\boldsymbol{r}')d\boldsymbol{r}'=1/\det\tilde{W}(t,\boldsymbol{r})$. This is exactly the same expression as for non-diffusing particles. For the moments of the concentration one finds

$$\langle n^\alpha\rangle=\int d\boldsymbol{\rho}_i\exp[-(\alpha-1)\sum\rho_i]P(t,\rho_i) \qquad (2.20)$$

with $P(t,\rho_i)$ given by Eq. (2.4). Note that the growth function $\gamma_{\rm L}(\alpha)$ in the Lagrangian frame is obtained by a mere shifting of the argument of the Eulerian growth function:

$$\langle n^\alpha(q(t,\boldsymbol{r}),t)\rangle=\langle n^{\alpha+1}(\boldsymbol{r},t)\rangle\,. \qquad (2.21)$$

Integral (2.20) can be calculated using the saddle-point approximation. The saddle-point value of $\tilde{\rho}_d$ given by $-c_\alpha t$, where c_α is an $\alpha-$dependent constant. i.e. the condition of applicability is $\kappa\int_0^t dt'\exp[2c_\alpha t']\ll r_{\rm v}^2$. Using convexity of the entropy one can show that large negative moments have negative c_α and therefore the above condition becomes time-independent. This can be simply seen noting that the averaged quantity $\exp[-\alpha\sum\tilde{\rho}_i]$ favors positive $\tilde{\rho}_i$ at negative α. Therefore, the diffusionless result is always correct for large negative moments. Diffusion cannot stop the formation of void regions with few particles inside.

On the other hand, from expression (2.20) one can see that for $\alpha>0$ any growing moment of n must be determined by a positive c_α, otherwise $\sum\tilde{\rho}_i>\tilde{\rho}_d>0$. Generally, one can assert the existence of the boundary $\alpha_{\rm b}$, such that $-\infty<\alpha_{\rm b}<1$. For $\alpha<\alpha_{\rm b}$, the saddle-point c_α is negative and the corresponding moment behaves as in the diffusionless case for all times, whereas for $\alpha>\alpha_{\rm b}$ the diffusionless approximation breaks down at large times. Since the moments with $\alpha<\alpha_{\rm b}$ are determined by the configurations on which the $r_{\rm d}$-volume is compressed backwards in time, one expects that $\alpha_{\rm b}$ is a monotonically increasing function of the velocity compressibility (as measured by $\sum\lambda_i$). For example, in the framework of the Kraichnan model one has $\alpha_{\rm b}=(\Gamma-4d)/(2\Gamma(d+2))$. We will refer to the $\alpha_{\rm b}<0$ case as the weakly compressible case, and $0<\alpha_{\rm b}<1$ as the strongly compressible one. It can be verified that this corresponds to the cases of $\tilde{\lambda}_d<0$ and $\tilde{\lambda}_d>0$ respectively. The same is valid for an arbitrary velocity statistics.

In fact at any time t one can consider the contribution of diverging configurations, so to say, the "ideal fluid contribution":

$$\langle n^\alpha\rangle_{\rm id}=\int_{\tilde{\rho}_d>0}d\rho_i\exp\left[-(\alpha-1)\sum\tilde{\rho}_i-tS\right]\,. \qquad (2.22)$$

Since the smallest size cannot be smaller than $r_{\rm d}$, it is necessary to introduce here the cutoff at $\tilde{\rho}_d=0$. It is clear that $\langle n^\alpha\rangle>\langle n^\alpha\rangle_{\rm id}$. Integral (2.22) has exponential time-dependence, $\langle n^\alpha\rangle_{\rm id}\propto\exp[\gamma_{\rm id}(\alpha)t]$. Note that due to the constraint $\tilde{\rho}_d>0$ the growth function $\gamma_{\rm id}$ is different from $\gamma(\alpha)$. For $\alpha<\alpha_{\rm b}$ the saddle-point is inside the domain of integration at all times. On the contrary, for $\alpha>\alpha_{\rm b}$, integral (2.22) is determined by the boundary, $\tilde{\rho}_d=0$.

In the weakly compressible situation, $\alpha_{\rm b}<0$, one has $\gamma_{\rm id}(\alpha_{\rm b})=\gamma(\alpha_{\rm b})>0$. From the continuity of $\gamma_{\rm id}(\alpha)$ we conclude that $\gamma_{\rm id}(\alpha)>0$ for $\alpha<\alpha'_{\rm b}<0$, where $\alpha'_{\rm b}$ is defined as $\gamma_{\rm id}(\alpha'_{\rm b})=0$. The inequality $\alpha'_{\rm b}<0$ follows from $\gamma_{\rm id}(\alpha)<\gamma(\alpha)$ and $\gamma(0)=0$. Therefore, the moments of the order $\alpha<\alpha'_{\rm b}<0$ satisfy $\langle n^\alpha\rangle>\exp(\gamma'_\alpha t)$ with positive γ'_α at all times (in fact, asymptotically the equality holds as mixing configurations can only lead to a growth slower than exponential, see below). It means that these moments become infinite in the steady state, which corresponds to the formation of the power-law asymptotic behavior for the PDF of the concentration near $n=0$: $\mathcal{P}(n)\propto n^{-\alpha'_{\rm b}-1}$. Diffusion does modify the growth of the moments with $\alpha_{\rm b}<\alpha<\alpha'_{\rm b}$, but the time dependence remains exponential.

In the strongly compressible case, $\alpha_{\rm b}>0$, one can immediately conclude that the moments with $\alpha<0$ grow exponentially with the ideal fluid exponents. For $\alpha>0$ the inequality $\langle n^\alpha\rangle>\langle n^\alpha\rangle_{\rm id}$ leads to no interesting conclusions at large times. The asymptotic behavior of the concentration PDF at $n\to0$ is $\mathcal{P}(n)\propto n^{-1}$.

Let us now consider the moments to which the main contribution is made by the contracting configurations (i.e. the moments with $\alpha>1$). The cutoff time t^* for the ideal growth is determined from the condition $\exp(2c_\alpha t^*_\alpha)\sim(r_{\rm v}/r_{\rm d})^2$, which gives $t^*_\alpha=c_\alpha^{-1}\ln(r_{\rm v}/r_{\rm d})$. This expression is exact in the limit of large Schmidt numbers. Note that the cutoff time depends on the order of the moment. For a quadratic in α entropy (e.g. for the Kraichnan model), c_α is a linear function of α. The steady-state dependence of the moments on Schmidt number can be estimated from below by $(r_{\rm v}/r_{\rm d})^{-\gamma(\alpha)/c_\alpha}$. One can expect that the dependence of the steady-state moments on the Schmidt number is linear for large α, since $\gamma_\alpha\sim\alpha c_\alpha$. This can be seen from the saddle-point expression for the moment. The linear dependence signifies less intermittent tail of the PDF as compared to the evolution problem.

III. SATURATION OF GROWTH DUE TO DIFFUSION

To analyze the behavior of the moments at larger times one should distinguish two cases: the $Re \sim 1$ case when the velocity correlation length L is of the order of $r_{\rm v}$, and the case of $Re \gg 1$, when $L \gg r_{\rm v}$. In the first case, advection becomes equivalent to the usual diffusion at the scales larger than $r_{\rm v}$. The moments get saturated and are given by the corresponding power of $r_{\rm v}/r_{\rm d}$. In the large Re case the velocity divergence is correlated at scales much larger than $r_{\rm v}$. The correlation between different viscous domains decays as a power of the distance between them. The configurations that coherently bring together different viscous domains determine the moments at this stage of evolution.

The moments continue to grow (in a power-law fashion) only for a particular case when the compressible correction to the velocity has the same scaling as the incompressible velocity. Since the compressible part is proportional to $(u\nabla)u$, then u and v have the same scaling only for a smooth velocity, $\delta u \propto r$, which has been studied in Sec. II. Note that up to logarithmic corrections this is true for a vorticity 2d cascade as well. However, for the turbulent velocity in the energy cascade, the velocity u is non-smooth, hence the compressible part has a different scaling. For example, the Kolmogorov phenomenology gives $\delta u \propto r^{1/3}$), so that $\delta v \propto r^{-1/3}$. It means that the compressibility is most important at small (viscous) scale so that the growth has to saturate and the level of fluctuations should not depend on the Reynolds number.

Unfortunately we still lack the formalism to describe Lagrangian statistics in the inertial interval with the same degree of universality as in the viscous interval. Nevertheless, to understand the most essential properties of the concentration fluctuations, one can use the simplest velocity statistics. We assume that the velocity is statistically isotropic, Gaussian, and has zero correlation time [16]. The pair-correlation function is given by

$$\langle v_\alpha(t,\boldsymbol{r},)v_\beta(0,0)\rangle = \delta(t)[V_0\delta_{\alpha\beta} - \mathcal{K}_{\alpha\beta}(\boldsymbol{r})]\,, \tag{3.1}$$
$$(d-1)\mathcal{K}_{\alpha\beta} = \left[\frac{(r^{d+1}u)'}{r^d} - c\right] r^2\delta_{\alpha\beta} - \left[\frac{(r^2u)'}{r} - c\right] r_\alpha r_\beta\,.$$

Note that $c = 0$ for incompressible flows.

We will assume that u and c have a regular expansion at $r \ll r_{\rm v}$, that is $u(r) \approx u(0) + u''(0)r^2/2 + \ldots$, and $c \approx c(0) + c''(0)r^2/2 + \ldots$. In the intermediate region $r_{\rm v} \ll r \ll L$, the functions u and c behave in power-law manners. However, due to relation (1.1), the scaling exponents of u and c are generally different. At large scales, $r \gg L$, the function u scales as r^{-2}. To guarantee that the variance of the velocity is positive, c must vanish faster than r^{-2} at $r \gg L$.

A convenient measure of compressibility is given by the ratio $\epsilon = c/u$. We will denote by ϵ_0 the value of ϵ for the smooth velocity (u, c constants). Despite all these crude simplifications the model enables to see the most interesting features of the growth and is used to illustrate the conclusions we believe to be model-independent.

A. The two-point correlation function

In this subsection we study the two-point correlation function of the concentration, $f(\boldsymbol{r}) = \langle n(0)n(\boldsymbol{r})\rangle$. In the framework of the model (3.1) it satisfies the closed equation

$$\partial_t f = \nabla_\alpha\nabla_\beta\left(\mathcal{K}_{\alpha\beta}f\right) + 2\kappa\nabla^2 f. \tag{3.2}$$

Let us first show that the dynamics of f is relaxational and then find the stationary solution to which it converges. For this purpose we must analyze the spectrum of the differential operator on the right-hand side of Eq. (3.2). The eigen functions must be regular at small distances. The boundary condition at large distances follows from the fact that the correlation function tends to a constant, equal to 1 (n_0^2 in the dimensional units). Therefore one must require that the eigen functions do not grow at infinity. The operator has the form of a $d-$dimensional Fokker-Planck operator, so that one can expect that it has no positive eigen values. Due to spherical symmetry of f on can rewrite Eq. (3.2) in the spherical coordinates

$$\partial_t f = r^{1-d}\hat{\mathcal{L}}_{FP}\left(r^{d-1}f\right)\,, \tag{3.3}$$
$$\hat{\mathcal{L}}_{FP} = \partial_r\left[r^2u + 2\kappa\right]e^{-\Phi}\partial_r e^{\Phi}\,, \tag{3.4}$$
$$e^{\Phi} = \frac{1}{r^{d-1}}\exp\left[\int\frac{rc(r)\,dr}{r^2u(r)+2\kappa}\right]\,. \tag{3.5}$$

Note that $\hat{\mathcal{L}}_{FP}$ has the form of a one-dimensional Fokker-Planck operator. Now we can show that the operator $\hat{\mathcal{L}}$ on the right-hand side of Eq. (3.2) has no positive eigen values, i.e. all the eigen functions of the operator satisfy $\hat{\mathcal{L}}f_E = -Ef_E$ with $E > 0$. Indeed, the evolution operator becomes proportional to a Laplacian at $r \gg L$. Therefore the negative energy eigen functions have exponential behavior at infinity. The boundary condition at infinity ensures that only exponentially decaying solutions are allowed. To show that for such solutions the boundary condition at $r = 0$ cannot be satisfied, we write the identity

$$\int_0^\infty dr\, f_E\, e^{\Phi}r^{2d-2}\hat{\mathcal{L}}f_E = -E\int_0^\infty dr\, f_E^2\, e^{\Phi}r^{2d-2}$$
$$= -\int_0^\infty dr\,\left(r^2u + 2\kappa\right)e^{-\Phi}\left(\partial_r e^{\Phi}r^{d-1}f_E\right)^2 < 0\,,$$

which proves that $E > 0$. We used integration by parts, which is possible only for functions decaying at infinity.

Let us now show that in fact the spectrum is continuous, covering the interval $[0,\infty)$. For definiteness we assume that $r_{\rm d}\ll r_{\rm v}\ll L$. Let us consider the equation at $r\ll r_{\rm v}$ where $u=u(0)$ and $c=\epsilon u(0)$:

$$(1+x^2)f''+\left[(d+1+\epsilon)x+\frac{d-1}{x}\right]f'+(d\epsilon+\lambda)f=0\,.$$

Above we have assumed that the distance is measured in units $r_{\rm d}$, where $r_{\rm d}^2=2\kappa/u(0)$. We have also introduced $\lambda=E/u(0)$. The solution of this equation satisfying the correct boundary conditions is

$$f_E(r)=C(E)F\left(\frac{d+\epsilon-\mu}{4},\frac{d+\epsilon+\mu}{4},\frac{d}{2},-\left(\frac{r}{r_{\rm d}}\right)^2\right),$$

where $\mu=\sqrt{(d-\epsilon)^2-4\lambda}$. Next, let us consider the inertial interval. Considering for simplicity $u=Dr^{-\gamma}$ and $c=\epsilon_\gamma Dr^{-\gamma}$ we find

$$r^2f''+(d+1-\gamma+\epsilon)rf'+\left[(d-\gamma)\epsilon+\frac{Er^\gamma}{D}\right]f=0\,,$$

The solution of this equation is

$$f=C_1(E)r^{-(d+\epsilon-\gamma)/2}J_\nu\left(\frac{2\sqrt{E}}{\gamma\sqrt{D}}r^{\gamma/2}\right)$$
$$+C_2(E)r^{-(d+\epsilon-\gamma)/2}N_\nu\left(\frac{2\sqrt{E}}{\gamma\sqrt{D}}r^{\gamma/2}\right), \qquad (3.6)$$

where $\nu=(d-\gamma-\epsilon)/\gamma$. Finally, at $r\gg L$ one can use the function (3.6) substituting $\gamma=2$, $\epsilon=0$ and V_0 instead of D

$$f=C_3(E)\,r^{1-d/2}J_{d/2-1}\left(\sqrt{\frac{E}{V_0}}r\right)$$
$$+C_4(E)\,r^{1-d/2}N_{d/2-1}\left(\sqrt{\frac{E}{V_0}}r\right),$$

We observe that unlike the case of $E<0$, for $E>0$ there is no additional restriction on the coefficients of the eigen function. Thus the matching problem can always be solved. The matching at $r\sim r_{\rm v}$ fixes the ratio of constants C_1/C_2, and the matching at $r\sim L_v$ fixes the ratio C_3/C_4. We conclude that the spectrum is positive continuous and non-degenerate. In fact, this property holds for any relation between $r_{\rm d}$ and $r_{\rm v}$.

Thus at large times f must converge to $f_0\equiv f_{\rm st}$. Note that decay at large times is prohibited by the inequality $f(t,0)=\langle n^2\rangle>\langle n\rangle^2=1$. Let us now find the stationary solution which satisfies

$$\partial_r\left(\left[r^2u+2\kappa\right]e^{-\Phi}\partial_r\left(e^{\Phi}r^{d-1}f_{\rm st}\right)\right)=0. \qquad (3.7)$$

The function must approach the square of the average concentration at infinity and be regular at zero. It is easy to see that the solution satisfying these conditions is the "zero flux" solution, proportional to $\exp[-\Phi]r^{1-d}$. It is given by

$$f_{\rm st}=\exp\left[\int_r^\infty\frac{r'c(r')dr'}{r'^2u(r')+2\kappa}\right]\,. \qquad (3.8)$$

The integral in Eq. (3.8) converges, because the function c decays faster than r^{-2} at $r\gg L$. Since $r^2u\sim V_0$ at $r\gg L$, we obtain the following asymptotic expression at these scales

$$f_{\rm st}-1\propto\left(\frac{L}{r}\right)^\alpha\,,$$

where we assumed that c decays as $r^{-2-\alpha}$, $\alpha>0$.

The behavior of $f_{\rm st}$ in the inertial interval, $r_{\rm v}\ll r\ll L$, crucially depends on whether u and c have the same scaling exponents. If they do, $f_{\rm st}(r)$ behaves as a negative power of r. The single point correlation function, $\langle n^2\rangle$, which can be estimated as $f_{\rm st}(r_{\rm d})$, is then proportional to a positive power of the Reynolds number. If, however, the velocity is not smooth in the inertial interval, u and c have different scaling exponents, and Eq. (3.8) shows that the main growth of $f_{\rm st}$ occurs below the viscous scale. Hence, $\langle n^2\rangle$ is independent of the Reynolds number. For example, for the Kolmogorov scaling, the solution (3.8) has the form $\ln f_{\rm st}\propto a^4r^{-4/3}r_{\rm v}^{-8/3}\beta^{-2}$ at $r\ll r_{\rm v}$.

Therefore, fluctuations of the concentration are mainly produced in the interval of scales $r\lesssim r_{\rm v}$, where the fluid velocity is smooth (i.e. in the viscous interval or in 2d vorticity cascade). One can write estimates

$$\langle n^2\rangle\simeq\left(\frac{r_{\rm v}}{r_{\rm d}}\right)^{\epsilon_0}\,,\quad \epsilon_0\simeq\beta^{-2}\left(\frac{a}{r_{\rm v}}\right)^4\,. \qquad (3.9)$$

Since $r_{\rm d}$ is by definition larger than a, significant fluctuations are possible only for very heavy particles with $\beta\approx 2\rho/3\rho_p<(a/r_{\rm v})^2\ln^{1/2}(r_{\rm v}/r_{\rm d})$.

To conclude, the fluctuations of concentration grow exponentially in any random compressible flow with a nonzero sum of Lyapunov exponents until this growth is restricted by finite-size effects (diffusion or discreetness of the particles). It is interesting to note that n can be considered as the density of the fluid itself, so that the finite-size effects are absent. Since density perturbations does not grow unlimited, one concludes that the phenomenon described here can take place only as a transient process when, for instance, large-Mach random flow with almost homogeneous density was initially created. In a stationary turbulence of the compressible fluid, the back reaction of density fluctuations on the flow stop the growth of the density fluctuations (we are indebted to V. Lebedev for this remark).

The work has been supported by NEC Research Institute, Minerva Foundation and Mitchell Research Fund.

We thank V. Lebedev and M. Vergassola for useful discussions.

[1] M.R. Maxey, J. Fluid Mech. **174**, 441 (1987).
[2] T. Elperin, I. Rogachevskii and N. Kleorin, Phys. Rev. Lett. **77**, 5373 (1996).
[3] V. Klyatskin and A. Saichev, JETP **84**, 716 (1997); V. Klyatskin and D. Gurarie, Sov. Ph. Usp. **169**, 171 (1999).
[4] M. Chertkov, I. Kolokolov and M.Vergassola, Phys. Rev. Lett. **80**, 512 (1997); Phys. Rev. E **56**, 5483 (1997).
[5] R. Ellis, *Entropy, Large Deviations and Statistical Mechanics*, (Springer-Verlag, Berlin, 1985).
[6] E. Balkovsky and A. Fouxon, Phys. Rev. E **60**, 4164 (1999).
[7] I. Goldhirsch, P-L. Sulem, and S. A. Orszag, Physica D **27**, 311 (1987).
[8] A. Gruzinov, S. Cowley and R. Sudan, Phys. Rev. Lett. **77**, 4342 (1996).
[9] V. I. Oseledec, Trans. Moscow Math. Soc. **19**, 197 (1968).
[10] M. Chertkov, G. Falkovich, I. Kolokolov and M. Vergassola, Phys. Rev. Lett. **83**, 4065 (1999).
[11] E. Balkovsky, A. Fouxon, V. Lebedev, Phys. Rev. Lett, **84**, 4765 (2000).
[12] Ya. B. Zeldovich, A. A. Ruzmaikin, S. A. Molchanov, and D. D. Sokolov, J. Fluid Mech. **144**, 1 (1984).
[13] M. Chertkov et al., Phys. Rev. E **51**, 5609 (1995).
[14] M. Chertkov, A. Gamba, and I. Kolokolov, Phys. Lett. A **192**, 435 (1994).
[15] K. Gawedzky and M. Vergassola, Physica D **138**, 63 (2000).
[16] B. Shraiman and E. Siggia, Nature **405**, 639 - 646 (2000).